Question 1. For what values of $a$ and $b$ will
\[ g(x) = \begin{cases} \ ax + b & x \leq -1 \\ ax^3 + x + 2b & x > -1 \end{cases} \]
be differentiable for all values of $x$?

Answer 1. To be differentiable, $g(x)$ must also be continuous at every point, and in particular continuous at $x = -1$. I.e.,
\[
\lim_{x \to -1^-} ax + b = \lim_{x \to -1^+} ax^3 + x + 2b
\]
\[ b - a = 2b - a - 1 \]
Hence $b = 1$. Next, we will check differentiability. Notice that, if derivative exists, then
\[
\lim_{h \to -1^-} f'(x) = \lim_{h \to -1^+} f'(x)
\]
\[ a = 3a + 1. \]
Thus, $a = -1/2$.

Question 2. Find the shortest distance from a given point $(0, b)$ on the $y$-axis to the parabola $x^2 = 4y$.

Answer 2. The distance between $(0, b)$ and an arbitrary point $(x, x^2/4)$ is given by $d = \sqrt{x^2 + (\frac{x^2}{4} - b)^2}$. It is sufficient to minimize distance square $d^2$ for simplicity.
\[
(x^2 + (\frac{x^2}{4} - b)^2)' = 0
\]
\[ 2x + \frac{x^3}{4} - xb = 0
\]
\[ x(2 - b + \frac{x^2}{4}) = 0
\]
Hence, the critical points are $\{0, -2\sqrt{b - 2}, 2\sqrt{b - 2}\}$. When substituted in the distance function, we get
\[
d^2((0, b), (0, 0)) = b^2
\]
\[
d^2((0, b), (-2\sqrt{b - 2}, b - 2)) = 4(b - 2) + 4 = 4b - 4
\]
\[
d^2((0, b), (2\sqrt{b - 2}, b - 2)) = 4(b - 2) + 4 = 4b - 4
\]
The minimum of these numbers depends on $b$, as when $b^2 - 4b + 4 \leq 0$ will imply that shortest distance is $b$ when $b \leq 2$ and shortest distance is $\sqrt{4b - 4}$ when $b \geq 2$. To sum up, the minimum distance is given by
\[
\text{min. distance} = \begin{cases} |b|, & b \leq 2, \\ 2\sqrt{b - 1}, & b \geq 2. \end{cases}
\]

Question 3. Find the area of the region in the first quadrant bounded on the left by the $y$-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$. 

Question 3.

From the graph, we can conclude that the area of the shaded region is

\[
\text{Area} = \int_0^1 2\sqrt{y} \, dy + \int_1^2 (3 - y - (y - 1)^2) \, dy
\]

\[
= \frac{4}{3}y^{3/2}\bigg|_0^1 + (2y + \frac{y^2}{2} - \frac{y^3}{3})\bigg|_1^2
\]

\[
= \frac{4}{3} + \frac{10}{3} - \frac{13}{6} = \frac{5}{2}.
\]

Question 4.

a) Find the length of the curve \( y^3 = x^2 \) from \((-1, 1)\) to \((1, 1)\).

b) Find the derivative

\[
\frac{d}{dx}\left( x^2 \int_0^x \sin t \, dt \right)
\]

Answer 4.

a) The graph of \( y^3 = x^2 \) is symmetric with respect to \( y \)-axis so the length asked is equal to 2 times of the length from \((0, 0)\) to \((1, 1)\). When we solve for \( x \), we get \( x = \frac{1}{2}y^{1/2} \) and its derivative is \( \frac{3}{2}y^{1/2} \). Therefore the length of the curve from \((0, 0)\) to \((1, 1)\) is given by

\[
\int_0^1 \sqrt{1 + \frac{9}{4}y^2} \, dy = (1 + \frac{9}{4}y^{3/2})^{3/2} \cdot \frac{2}{3} \cdot \frac{4}{9}
\]

\[
= \frac{8}{27} (\sqrt{\frac{13}{8}} - 1)
\]

and the final result will be

\[
2 \cdot \frac{8}{27} (\sqrt{\frac{13}{8}} - 1) = \frac{16}{27} (\frac{\sqrt{13}}{8} - 1).
\]

b) \( 2x \int_0^x \frac{\sin t}{t} \, dt + x^2 \cdot \frac{\sin x^2}{x^2} \cdot 2x = 2x \sin x^2 + 2x \int_0^x \frac{\sin t}{t} \, dt. \)

Question 5. Find the volume of the solid generated by revolving the region between the \( y \)-axis and the curve \( x = 12(y^2 - y^3) \) about the \( x \)-axis.

Answer 5. Using shell method, the volume is given by

\[
\int_0^1 2\pi \cdot y \cdot 12(y^2 - y^3) \, dy
\]

\[
= 6\pi y^4 - \frac{24\pi}{5} y^5 \bigg|_0^1 = \frac{6\pi}{5}.
\]