MATH 264 PROBLEM HOMEWORK #2

Due to December 9, 2004@17:30

PROBLEMS

1. (a) A gambler has in his pocket a fair coin and a two-headed coin. He selects one of the coins at random; when he flips it, it shows heads. What is the probability that it is the fair coin?

(b) Suppose that he flips the same coin a second time and again it shows heads. What is the probability that it is the fair coin?

(c) Suppose that he flips the same coin a third time and it shows tails. What is now the probability that it is the fair coin?

Solution. It is to see from the following tree diagram

\[
\begin{array}{c}
\frac{1}{2} \\
\text{fair coin} \\
\frac{1}{2} \\
\text{two-headed coin}
\end{array}
\]

<table>
<thead>
<tr>
<th></th>
<th>head</th>
<th>head</th>
<th>tail</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
</tbody>
</table>

that

(a) \[ P(\text{fair coin} \mid \text{head}) = \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = \frac{1}{3}. \]

(b) \[ P(\text{fair coin} \mid \text{head-head}) = \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}} = \frac{\frac{1}{8}}{\frac{1}{8} + \frac{1}{8}} = \frac{1}{5}. \]

(c) \[ P(\text{fair coin} \mid \text{head-head-tail}) = \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}} = \frac{\frac{1}{16}}{\frac{1}{16} + \frac{1}{16} + \frac{1}{16}} = \frac{1}{3}. \]

2. A fair die is rolled three times and the random variable \(X\) records the number of different outcomes that result. For example, if two 5’s and one 4 are rolled, then \(X\) records two different outcomes. Determine

(a) the probability distribution of \(X\),

(b) the expected value of \(X\),

(c) the variance of \(X\).

Solution. It is clear that \(X\) can take the values 1, 2, or 3.

More precisely, \(X\) takes the value 1 when we have observed

\[ \{X = 1\} = \{1-1-1, 2-2-2, \ldots, 6-6-6\} \]

on three tosses,

\(X\) takes the value 2 when we have observed

\[ \{X = 2\} = \{1-1-2, 1-2-1, 2-1-1, 2-2-1, 2-1-2, 1-2-2, \ldots, 5-5-6, 5-6-5, 6-5-5, 6-6-5, 6-5-6, 5-6-6\} \]

on three tosses, and

\(X\) takes the value 3 when we have observed

\[ \{X = 3\} = \{1-2-3, 1-3-2, 2-1-3, 2-3-1, 3-1-2, 3-2-1, \ldots, 4-5-6, 4-6-5, 5-4-6, 5-6-4, 6-5-4, 6-4-5\} \]
on three tosses.

It is clear that $|S| = 6 \cdot 6 \cdot 6 = 216$,

$|X = 1| = \binom{6}{1} = 6$, $|X = 2| = \binom{6}{2} \cdot 2 \cdot \frac{3!}{2!1!} = \frac{6 \cdot 5 \cdot 2 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 1} = 15 \cdot 2 \cdot 3 = 90$, and

$|X = 3| = \binom{6}{3} \cdot 3! = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 20 \cdot 6 = 120$.

Thus,

(a) $X$ has the probability distribution given by the table

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{6}{216}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{90}{216}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{120}{216}$</td>
</tr>
</tbody>
</table>

(b) Using the formula $\mu = \sum x f(x)$, we obtain that

$\mu = 1 \cdot \frac{6}{216} + 2 \cdot \frac{90}{216} + 3 \cdot \frac{120}{216} = \frac{91}{36} \approx 2.53$.

(c) Using the formula $\sigma^2 = (\sum x^2 f(x)) - \mu^2$, we obtain that

$\sigma^2 = \left(1 \cdot \frac{6}{216} + 4 \cdot \frac{90}{216} + 9 \cdot \frac{120}{216}\right) - \left(\frac{91}{36}\right)^2 = \frac{395}{1296} \approx 0.30$.

3. A dozen urns each contain four red marbles and seven green ones (all 132 marbles are of the same size). If a dozen students each select a different urn and then draw (with replacement) five marbles, what is the probability that at least one student draws at least one red marble?

**Solution.**

For each student, the probability that s/he only selects the green balls (in five trials) is

$$\left(\frac{7}{11}\right)^5.$$  

Thus, the probability that all students only select the green balls (in five trials) is

$$\left(\left(\frac{7}{11}\right)^5\right)^{12} = \left(\frac{7}{11}\right)^{60},$$

Hence, the probability that at least one student draws at least one red marble is

$$1 - \left(\frac{7}{11}\right)^{60} = 0.0918.$$

4. From past experience it is felt that, the age of the mother at the birth of her first child, is normally distributed with a mean of 20 years and a variance of 9, world-wide. Find the probability that a randomly selected mother has her first child

(a) before age 16,
(b) after age 30,
(c) between the ages of 16 and 24.

**Solution.** Let $X$ be the age of the mother at the birth of her first child. Then $X$ is normally distributed with $\mu = 20$ and $\sigma^2 = 9$ (that is, $\sigma = 3$).

(a) We are asked to find $P(X < 16)$. Since $\frac{X - \mu}{\sigma} = \frac{X - 20}{3}$ is the standard normal distribution, we have

$$P(X < 16) = \left(\frac{X - 20}{3} < \frac{16 - 20}{3}\right) = \left(\frac{X - 20}{3} < -1.33\right) = 0.5 - \left(-1.33 < \frac{X - 20}{3} < 0\right) = 0.5 - 0.4082 = 0.0918.$$
(b) We are asked to find $P(X > 30)$. Similarly, we have

$$P(X > 30) = \left( \frac{X - 20}{3} > \frac{30 - 20}{3} \right) = \left( \frac{X - 20}{3} > 3.33 \right)$$

$$= 0.5 - \left( 0 < \frac{X - 20}{3} < 3.33 \right) = 0.5 - 0.49957 = 0.00043.$$ 

(c) We are asked to find $P(16 < X < 24)$, and, we have

$$P(16 < X < 24) = \left( \frac{16 - 20}{3} < \frac{X - 20}{3} < \frac{24 - 20}{3} \right) = \left( -1.33 < \frac{X - 20}{3} < 1.33 \right)$$

$$= 2 \cdot \left( 0 < \frac{X - 20}{3} < 1.33 \right) = 2 \cdot 0.4082 = 0.8164.$$